On a Hamiltonian form of an elliptic spin Ruijsenaars-Schneider system

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1 Introduction

An elliptic Ruijenaars-Schneider (RS) model [1] is a Hamiltonian system of N interacting particles with a Hamiltonian

$$H = \sum_{j=1}^{N} e^{p_j} \prod_{s \neq j}^{N} \left(\frac{\sigma(x_j - x_s + \eta)\sigma(x_j - x_s - \eta)}{\sigma^2(x_j - x_s)} \right)^{1/2} \tag{1}$$

and the canonical symplectic form $\omega = \sum \delta p_i \wedge \delta x_i$, where $p_i = \dot{x}_i$.

The equations of motions are

$$\ddot{x}_i = \sum_{s \neq i} \dot{x}_i \dot{x}_s (V(x_s - x_i) - V(x_i - x_s)), \tag{2}$$

where $V(x) = \zeta(x + \eta) - \zeta(x)$, and $\zeta(x)$ is a Weierstrass zeta function.

The limit when one or two periods of the elliptic curve go to infinity yields a trigonometric or rational system. A RS system is a relativistic generalization of the Calogero-Moser model.

A spin generalization of RS system was suggested in [2]. Each particle additionally carries two l-dimensional vectors a_i and b_i that describe the internal degrees of freedom and affect the interaction. Remarkably, the equations of motion remain integrable and are given by the formulas

$$\begin{cases} \dot{f}_{ij} = \sum_{k \neq j} f_{ik} f_{kj} V(x_j - x_k) - \sum_{k \neq i} f_{ik} f_{kj} V(x_k - x_i) \\ \dot{x}_i = f_{ii}, \end{cases} \tag{3}$$

where $f_{ij} = b_i^T a_j$.

It was shown in [3] using the universal symplectic form (proposed in [4]) that a spin elliptic RS system is Hamiltonian. An expression of a symplectic form (or Poisson structure) in explicit coordinates is known only in the rational and trigonometric limits (see [7]).

The aim of this paper is to compute ω in the original coordinates x_i and f_{ij} in the simplest elliptic case of 2 particles, $N = 2$. We compare the obtained 2-form with a symplectic form for a system without spin and with a Poisson structure found in [7] in the rational case.

2 Symplectic form in the case $N = 2$

The general procedure developed by Krichever and Phong in [4] allows to construct action-angle variables for an elliptic RS system and its spin generalization. It was done in [3]. The upshot of the procedure is the following.

A Lax representation with a spectral parameter for an elliptic RS system has been found in [2]. A Lax matrix is

$$L_{ij} = f_i \Phi(x_i - x_j - \eta), \text{ where } \Phi(x, z) = \frac{\sigma(z + x + \eta)}{\sigma(z + \eta)\sigma(x)} \left[\frac{\sigma(z - \eta)}{\sigma(z + \eta)} \right]^{x/2\eta}. \quad (4)$$

The spectral parameter z is defined on an elliptic curve Γ_0 with a cut between points $z = \eta$ and $z = -\eta$.

The universal symplectic form is given by the formula

$$\omega = -\frac{1}{2} \sum_{q \in I} \operatorname{res}_q \operatorname{Tr} \left(\Psi^{-1} L^{-1} \delta L \wedge \delta \Psi - \Psi^{-1} \delta \Psi \wedge K^{-1} \delta K \right) dz, \quad (5)$$

where the sum is taken over the poles of L and zeroes of $det\, L$. Ψ is a matrix composed of eigenvectors of L, which has poles $\hat{\gamma}_s$ on the spectral curve $\hat{\Gamma}$: $det\,(L - kI) = 0$ due to normalization of eigenvectors. k is a meromorphic function on $\hat{\Gamma}$ and the matrix $K = diag(k_1, ..., k_N)$ is composed of values of k on different sheets of $\hat{\Gamma}$.

ω doesn't depend on the gauge transformations $L \to gLg^{-1}$ and the normalization of eigenvectors on the leaves where the form $\delta \ln k dz$ is holomorphic. $\operatorname{Tr}(...)\, dz$ is a meromorphic differential, and the sum of all its residues is zero. Using these facts, one can show that on the leaves

$$\omega = \sum_s \delta \ln k(\hat{\gamma}_s) \wedge \delta z(\hat{\gamma}_s). \quad (6)$$

Computations performed in [3] for Lax matrix (4) show that

$$\omega = \sum_i \delta \ln f_i \wedge \delta x_i + \sum_{i \neq j} V(x_i - x_j) \delta x_i \wedge \delta x_j, \quad (7)$$

where

$$f_i = e^{p_i} \prod_{s \neq i}^{N} \left(\frac{\sigma(x_i - x_s + \eta)\sigma(x_i - x_s - \eta)}{\sigma^2(x_i - x_s)} \right)^{1/2}$$

and the Hamiltonian for system (2) is $H = \sum_{i=1}^{N} f_i$.

A Lax representation with a spectral parameter for an elliptic spin RS system (3) has been found in [2]. The Lax matrix is $L_{ij} = f_{ij} \Phi(x_i - x_j - \eta)$. Formally, equations (3) are Hamiltonian with $H = \sum_{i=1}^{N} f_{ii}$ and symplectic form (5) (see [3] for details). The goal of this paper is to compute form (5) in the original coordinates x_i and f_{ij}.

After the gauge transformation by a diagonal matrix $g = diag(\Phi(x_1, z), \lambda\Phi(x_2, z))$ with an appropriate choice of λ, the matrix L_{ij} becomes

$$L = \begin{pmatrix} -f_1\dfrac{\sigma(z)}{\sigma(\eta)} & f_3\dfrac{\sigma(z-x_1+x_2)\sigma(z+x_1+\eta)\sigma(x_2)}{\sigma(x_2-x_1-\eta)\sigma(z+x_2+\eta)\sigma(x_1)} \\ f_3\dfrac{\sigma(z+x_1-x_2)\sigma(z+x_2+\eta)\sigma(x_1)}{\sigma(x_1-x_2-\eta)\sigma(z+x_1+\eta)\sigma(x_2)} & -f_2\dfrac{\sigma(z)}{\sigma(\eta)} \end{pmatrix} \frac{1}{\sqrt{\sigma(z+\eta)\sigma(z-\eta)}},$$

where $f_1 \equiv f_{11}, f_2 \equiv f_{22}$ and $f_3 \equiv \sqrt{f_{12}f_{21}}$.

The matrix L is defined on a curve Γ of genus $g = 2$, which is a 2-sheeted cover of the elliptic curve Γ_0 with 2 branch points $z = \eta$ and $z = -\eta$.

The spectral curve $\hat{\Gamma}$ of L is defined by the equation $R = det\,(L_{ij} - k) = 0$. It is a 2-sheeted cover of Γ, and the function $\partial_k R$ has 4 simple poles on $\hat{\Gamma}$ above points $z = \pm\eta$. $\partial_k R$ is a meromorphic function on $\hat{\Gamma}$, hence it also has 4 zeroes. Its zeroes are precisely the branch points of $\hat{\Gamma}$ over Γ, and the Riemann-Hurwitz formula implies that the genus of $\hat{\Gamma}$ is $\hat{g} = 5$.

The matrix valued differential Ldz can be seen as a global section of the bundle $End(V_{\gamma,\alpha}) \otimes \Omega^{1,0}(\Gamma)$. $V_{\gamma,\alpha}$ is a vector bundle determined by Tyurin parameters $z(\gamma_i) = -x_1 - \eta$, $z(\gamma_j) = -x_2 - \eta$, and $\alpha_i = (0,1)^T$, $\alpha_j = (1,0)^T$, where $i = 1,2$ and $j = 3,4$.

The set I in (5) is $I = \{\gamma_s, 0, \pm z_0\}$, where z_0 is defined by the equation $det\,L(z_0) = 0$, or

$$f_1 f_2 \frac{\sigma^2(z_0)}{\sigma^2(\eta)} - f_3^2 \frac{\sigma(z_0+x_1-x_2)\sigma(z_0-x_1+x_2)}{\sigma(x_1-x_2-\eta)\sigma(x_2-x_1-\eta)} = 0.$$

Notice, that we can use variables $(x_1, x_2, f_1, f_2, z_0)$ instead of $(x_1, x_2, f_1, f_2, f_3)$.

Theorem 1. *In the case $N = 2$ the elliptic spin RS system is Hamiltonian with a symplectic form*

$$\omega = -\delta \ln f_1 \wedge \delta x_1 - \delta \ln f_2 \wedge \delta x_2 + 2\tilde{V}(x_1 - x_2)\delta x_1 \wedge \delta x_2 \tag{8}$$

and Hamiltonian $H = f_1 + f_2$, where $\tilde{V}(x) = \zeta(x + z_0) - \zeta(x)$. The spinless case corresponds to $z_0 = \eta$.

Proof. The eigenvector ψ of L in any normalization is a meromorphic function on $\hat{\Gamma}$ and it has $\hat{g} + 1 = 6$ poles $\hat{\gamma}_s$. The proof of formula (6) in [6] assumes that the situation is in general position, i.e. projections of points $\hat{\gamma}_i$ don't coincide with γ_s.

Most appropriate normalization here is $\psi_1 \equiv 1$, because it easily allows us to find poles $\hat{\gamma}_s$ of ψ. Two of them $(s = 1,2)$ lie above the point $z = x_1 - x_2$, and the other are above $z = -x_1 - \eta$ $(s = 3,4,5,6)$. This is not the case of general position, but it turns out that the same formula (6) still holds.

The proof in [5] and [6] implies that 2-form (5) in the normalization $\psi_1 \equiv 1$ equals to $\omega_0 = \sum_{s=1}^{2} \delta \ln k(\hat{\gamma}_s) \wedge \delta z(\hat{\gamma}_s)$.

A change of normalization of Ψ from $\psi_1 \equiv 1$ to $\sum \psi_i \equiv 1$ (the last one is in "general position") corresponds to the transformation $\tilde{\Psi} = \Psi V$, where

$$V = \begin{pmatrix} \dfrac{L_{12}}{k_1 - L_{11} + L_{12}} & 0 \\ 0 & \dfrac{L_{12}}{k_2 - L_{11} + L_{12}} \end{pmatrix}.$$

According to the computations in [6],

$$\omega = \omega_0 + \sum_{q \in I} \operatorname{res}_q \operatorname{Tr}\left(K^{-1}\delta K \wedge \delta V V^{-1}\right) dz.$$

Since ω has to be restricted to the leaves where $\delta \ln k dz$ is holomorphic (which is equivalent to 2 conditions: $\delta\eta = 0$ and $\delta z_0 = 0$), the only non-zero residue in the second term is at the point $z(\gamma_i) = -x_1 - \eta$. After computing the residue, we get that $\omega = \omega_0 + \sum_{s=3}^{6} \delta \ln k(\hat{\gamma}_s) \wedge \delta z(\hat{\gamma}_s)$, i.e. effectively formula (6) holds in both normalizations.

Substituting $\hat{\gamma}_s$ in (6), we find that

$$\omega = -\delta \ln f_1 \wedge \delta x_1 - \delta \ln f_2 \wedge \delta x_2 + 2\tilde{V}(x_1 - x_2)\delta x_1 \wedge \delta x_2,$$

where $\tilde{V}(x) = \zeta(x + z_0) - \zeta(x)$.

The Hamiltonian $H = f_1 + f_2$ defines the flow

$$\begin{cases} \dot{f}_1 = -f_1 f_2(\zeta(z_0 + x_1 - x_2) - \zeta(z_0 - x_1 + x_2) - 2\zeta(x_1 - x_2)) \\ \dot{f}_2 = f_1 f_2(\zeta(z_0 + x_1 - x_2) - \zeta(z_0 - x_1 + x_2) - 2\zeta(x_1 - x_2)) \\ \dot{x}_1 = f_1 \\ \dot{x}_2 = f_2. \end{cases}$$

Using identities for Weierstrass σ-functions, namely,

$$\sigma(a+c)\sigma(a-c)\sigma(b+d)\sigma(b-d) - \sigma(a+d)\sigma(a-d)\sigma(b+c)\sigma(b-c) =$$

$$= \sigma(a+b)\sigma(a-b)\sigma(c+d)\sigma(c-d), \text{ and}$$

$$\zeta(a) + \zeta(b) + \zeta(c) - \zeta(a+b+c) = \frac{\sigma(a+b)\sigma(b+c)\sigma(a+c)}{\sigma(a)\sigma(b)\sigma(c)\sigma(a+b+c)},$$

it follows from the definition of z_0 that

$$\begin{aligned} f_1 f_2(2\zeta(x_1 - x_2) + \zeta(z_0 - x_1 + x_2) - \zeta(z_0 + x_1 - x_2)) &= \\ = f_3^2(2\zeta(x_1 - x_2) + \zeta(\eta - x_1 + x_2) - \zeta(\eta + x_1 - x_2)).& \end{aligned} \tag{9}$$

With the help of this identity, we can show that the above equations are equivalent to

$$\begin{cases} \ddot{x}_1 = f_3^2(2\zeta(x_1 - x_2) + \zeta(\eta - x_1 + x_2) - \zeta(\eta + x_1 - x_2)) \\ \ddot{x}_2 = -f_3^2(2\zeta(x_1 - x_2) + \zeta(\eta - x_1 + x_2) - \zeta(\eta + x_1 - x_2)), \end{cases}$$

which is an RS system.

The spinless case occurs when $f_3^2 = f_1 f_2$ and $z_0 = \eta$ as one can observe from (9). $\square$

Remark. A Poisson structure was found in [7] in the rational limit for arbitrary N (see formula (3.31) in [7]). In the case of 2 particles it is non-degenerate and defined on a 6-dimensional space $(f_{11}, f_{12}, f_{21}, f_{22}, x_1, x_2)$. The corresponding 2-form is defined on the same space and coincides with (8) on the leaves $\delta z_0 = 0$ and after reduction with respect to the action $f_{12} \to f_{12}/\lambda$, $f_{21} \to f_{21}\lambda$.

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